exactly

Chapter III

The Second fundamental form.

Sec.1 The Second fundamental form

Thm 1 If k_n is the normal curvature of a curve at a point on a surface, then $k_n = \dfrac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$

where $L = N\cdot r_{11}$, $M = N\cdot r_{12}$, $N = N\cdot r_{22}$ and E, F, G are the first fundamental coefficients.

Proof: Let $\bar{r}$ be the position vector of any point on the curve. If k_n is the normal curvature of the curve at P on a surface, ~~we know that~~

$$\bar{r}'' = k_n N + \lambda \bar{r}_1 + \mu \bar{r}_2 \longrightarrow (1)$$

Since $\bar{r}_1 \cdot N = 0$, $\bar{r}_2 \cdot N = 0$, taking dot product with N on both sides of (1),

$$k_n = \bar{r}'' \cdot N \longrightarrow (2)$$

Since $\bar{r}$ is a fun. of u & v, we have

$$\bar{r}' = \frac{\partial \bar{r}}{\partial u}\frac{du}{ds} + \frac{\partial \bar{r}}{\partial v}\cdot\frac{dv}{ds} = \bar{r}_1 u' + \bar{r}_2 v' \longrightarrow (3)$$

Differentiating (3) w.r.to s on both sides,

$$\bar{r}'' = \bar{r}_1 u'' + \bar{r}_2 v'' + (\bar{r}_1)' u' + (\bar{r}_2)' v'$$

But $(\bar{r}_1)' = \dfrac{\partial \bar{r}_1}{\partial u} u' + \dfrac{\partial \bar{r}_1}{\partial v} v' = \bar{r}_{11} u' + \bar{r}_{12} v' \rightarrow (5)$

$(\bar{r}_2)' = \dfrac{\partial \bar{r}_2}{\partial u} u' + \dfrac{\partial \bar{r}_2}{\partial v} v' = \bar{r}_{21} u' + \bar{r}_{22} v' \rightarrow (6)$

$\therefore \bar{r}'' = \bar{r}_1 u'' + \bar{r}_2 v'' + \bar{r}_{11} u'^2 + 2\bar{r}_{12} u'v' + \bar{r}_{22} v'^2$

Taking dot product with $\bar{N}$ on both sides of (?) and using $\bar{r}_1 \cdot \bar{N} = 0$, $\bar{r}_2 \cdot N = 0$ and (2) we have

$$N \cdot \bar{r}'' = k_n = (N \cdot \bar{r}_{11}) u'^2 + 2(N \cdot \bar{r}_{12}) u'v' + (N \cdot \bar{r}_{22}) v'^2$$

If $L = N \cdot r_{11}$, $M = N \cdot r_{12}$ and $N = N \cdot r_{22}$

we have $k_n = L u'^2 + 2M u'v' + N v'^2$

$$= \frac{L du^2 + 2M du dv + N dv^2}{ds^2}$$

Since $ds^2 = E du^2 + 2F du dv + G dv^2$, we have

$$k_n = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$$

<u>Defn.</u> The quadratic form $L du^2 + 2M du dv + N dv^2$ is called the second fundamental form of the surf. and L, M, N which are functions of u, v are called second fundamental coefficients.

<u>Thm. 2</u>

i) $L = -N_1 \cdot \bar{r}_1$, $M = -\bar{N}_1 \cdot r_2 = -N_2 \cdot r_1$

$N = -N_2 \cdot \bar{r}_2$

ii) $L = \dfrac{[\bar{r}_{11}, \bar{r}_1, \bar{r}_2]}{H}$

$M = \dfrac{[\bar{r}_{12}, \bar{r}_1, \bar{r}_2]}{H} = \dfrac{[\bar{r}_{21}, \bar{r}_1, \bar{r}_2]}{H}$

$N = \dfrac{[\bar{r}_{22}, \bar{r}_1, \bar{r}_2]}{H}$

Proof: At any point P on the surface $\bar{r}_1$ and $\bar{r}_2$ are tangential to the surface so that we have

$$\underbrace{\bar{N}\cdot\bar{r}_1 = 0}_{(1)} \;\&\; \bar{N}\cdot\bar{r}_2 = 0 \longrightarrow (2)$$

Differentiating (1) w.r.to 'u' we get

$$N\cdot\bar{r}_{11} + \bar{N}_1\cdot\bar{r}_1 = 0 \Rightarrow \bar{r}_{11}\cdot N = -\bar{r}_1\cdot N_1$$

Since $L = N\cdot\bar{r}_{11}$ we get $L = -\bar{N}_1\cdot\bar{r}_1$

Since ~~M =~~ Differentiating (1) $N\cdot\bar{r}_1 = 0$ w.r.to 'v' we get

$$N\cdot\bar{r}_{12} + N_2\cdot\bar{r}_1 = 0 \Rightarrow N\cdot\bar{r}_{12} = -N_2\cdot\bar{r}_1$$

Since $M = N\cdot\bar{r}_{12}$, $M = -N_2\cdot\bar{r}_1$

Differentiating (2) w.r.to 'u' we get

$$N\cdot\bar{r}_{21} + N_1\cdot\bar{r}_2 = 0 \Rightarrow N\cdot\bar{r}_{21} = -N_1\cdot\bar{r}_2$$

$$\therefore M = -\bar{N}_1\cdot\bar{r}_2$$

Differentiating (2) w.r.to 'v' we get

$$N\cdot\bar{r}_{22} + N_2\cdot\bar{r}_2 = 0 \rightarrow N\cdot\bar{r}_{22} = -N_2\cdot\bar{r}_2$$

Since $N = \bar{N}\cdot\bar{r}_{22}$, $N = -\bar{N}_2\cdot\bar{r}_2$

ii) Now

$$[\bar{r}_{11}, \bar{r}_1, \bar{r}_2] = \bar{r}_{11}\cdot(\bar{r}_1\times\bar{r}_2) \qquad [\because \bar{r}_1\times\bar{r}_2 = HN]$$
$$= \bar{r}_{11}\cdot(HN)$$
$$= H(N\cdot\bar{r}_{11})$$
$$= HL.$$

Thus $[\bar{r}_{11}, \bar{r}_1, \bar{r}_2] = HL \Rightarrow L = \frac{1}{H}[\bar{r}_{11}, \bar{r}_1, \bar{r}_2]$

Further $[\bar{r}_{12}, \bar{r}_1, \bar{r}_2] = \bar{r}_{12}\cdot(\bar{r}_1\times\bar{r}_2) = \bar{r}_{12}\cdot(HN) = H(N\cdot\bar{r}_{12}) = HM$

$$\therefore M = \frac{1}{H}[\bar{r}_{12}, \bar{r}_1, \bar{r}_2]$$

Now, $[\bar{r}_{22}, \bar{r}_1, \bar{r}_2] = \bar{r}_{22}\cdot(\bar{r}_1\times\bar{r}_2) = \bar{r}_{22}\cdot(HN) = H(N\cdot\bar{r}_{22}) = HN.$

$$\therefore N = \frac{1}{H}[\bar{r}_{22}, \bar{r}_1, \bar{r}_2]$$

Thm. Meusnier's Theorem:

If ϕ is the angle between the Principal normal $\bar{n}$ to a curve on a surface and the surface normal $\bar{N}$, then $k_n = k\cos\phi$.

Proof: We know that at any point on a curve of a surface

$$\bar{r}'' = k_n\bar{N} + \lambda\bar{r}_1 + \mu\bar{r}_2 \longrightarrow (1)$$

Taking dot product with $\bar{N}$ on both sides of (1)

we get $\quad k_n = \bar{r}''\cdot\bar{N} \longrightarrow (2)$

But we know that $\bar{r}'' = \dfrac{d\bar{t}}{ds} = k\bar{n} \longrightarrow (3)$

Using (3) in (2), $\quad k_n = k\bar{n}\cdot N = k\cos\phi$

Thus the normal curvature k_n is the projection on the surface normal $\bar{N}$ of a vector of length k along the principal normal to a curve.

Classification of points on a surface:-

The second fundamental form $Ldu^2 + 2Mdudv + Ndv^2$ is a quadratic form in du and dv.

The discriminant of the quadratic form is $LN - M^2$.

Using the discriminant, we can rewrite the second fundamental form as $Ldu^2 + 2Mdudv + Ndv^2$

$$= \frac{1}{L}\left[(Ldu + Mdv)^2 + (LN - M^2)dv^2\right]$$

The above quadratic form is positive definite, a perfect square or indefinite according as $LN - M^2 > 0$, $LN - M^2 = 0$ or $LN - M^2 < 0$. Consider the following three particular cases.

case (i) Let $LN - M^2 > 0$.

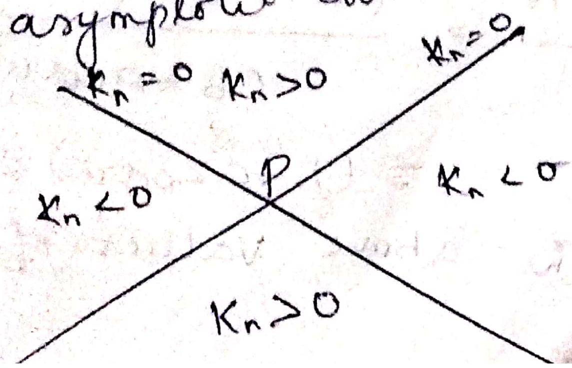
Since the discriminant is positive, the quadratic form is positive at any point P on the surface. Hence k_n has the same sign for all directions at P. In this case, the point P is called an elliptic point.

case (ii) Let $LN - M^2 = 0$.

Since $L \neq 0$, the quadratic form becomes $(L\,du + M\,dv)^2$. Hence k_n has the same sign for all directions through P except when $k_n = 0$ which implies $\frac{du}{dv} = -\frac{M}{L}$. In this case the point P is called a parabolic point. The critical direction are called asymptotic directions.

case (iii) Let $LN - M^2 < 0$.

The quadratic form is indefinite and k_n does not retain the same sign for all directions at P. It may happen that k_n is zero, positive or negative. If $k_n = 0$, the two directions corresponding to $k_n = 0$ form an angle with respect to the two roots (u_1, v_1) and (u_2, v_2) of the quadratic form. k_n is positive for certain direction lying inside the angular region and negative for directions outside this angular region. The point P is called a hyperbolic point. The two critical directions bounding the angle are called asymptotic directions.



Prm: Show that the anchor ring contains all types of points namely elliptic, parabolic and hyperbolic on the surface.

Soln: Any point on the surface of the anchor ring is $\bar{r} = [(b+a\cos u)\cos v,\ (b+a\cos u)\sin v,\ a\sin u]$

where $0 < u \le 2\pi$ & $0 \le v < 2\pi$

Using the sign of $LN-M^2$, we shall find the nature of points on the anchor ring.

$\bar{r}_1 = (-a\sin u\cos v,\ -a\sin u\sin v,\ a\cos u)$

$\bar{r}_2 = (-(b+a\cos u)\sin v,\ (b+a\cos u)\cos v,\ 0)$

$\bar{r}_1 \times \bar{r}_2 = (b+a\cos u)[-a\cos u\cos v,\ -a\sin v\cos u,\ -a\sin u]$

$E = r_1^2 = a^2 \quad F = \bar{r}_1 \cdot \bar{r}_2 = 0 \quad G = \bar{r}_2^2 = (b+a\cos u)^2$

& $H = a(b+a\cos u)$

$\bar{r}_{11} = (-a\cos u\cos v,\ -a\cos u\sin v,\ -a\sin u)$

$\bar{r}_{12} = (a\sin u\sin v,\ -a\sin u\cos v,\ 0)$

$\bar{r}_{22} = [-(b+a\cos u)\cos v,\ -(b+a\cos u)\sin v,\ 0]$

$$L = \frac{\bar{r}_{11} \cdot N}{H} = \frac{(b+a\cos u)}{a(b+a\cos u)} a^2 = a$$

$$M = \bar{r}_{12} \cdot N = 0$$

$$N = \frac{\bar{r}_{22} \cdot N}{H} = \frac{(b+a\cos u)^2 a\cos u}{(b+a\cos u)a}$$

$$= (b+a\cos u)\cos u$$

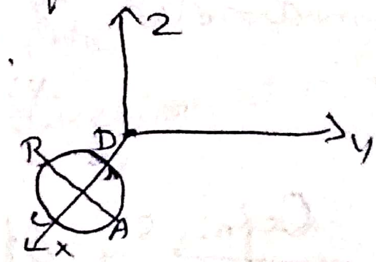
Using the above values of L, M & N we get

$$LN - M^2 = a(b + a\cos u)\cos u$$

Since $b > a$, $(b + a\cos u)$ is positive for all values of u in its domain. So, the sign of $LN - M^2$ is determined only by $\cos u$ alone. This leads to the following 3 cases of u.

Case i) Let $0 < u < \frac{\pi}{2}$ or $\frac{3\pi}{2} < u < 2\pi$.

Then $\cos u$ is positive and so $LN - M^2$ is positive. So all those points corresponding to these values of u are elliptic. The points on the torus corresponding to these values of u are at a distance greater than b from the axis of rotation. Hence the points on the torus whose distances from the axis of rotation are greater than b are elliptic points.



Case (ii) Let $u = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

Then $\cos u = 0 \Rightarrow LN - M^2 = 0$.

Hence all points for which $u = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ are parabolic. As v varies, the points lie on circles of radii b at the top and bottom of the surface obtained by rotating B and A.

Case (iii) Let $\frac{\pi}{2} < u < \frac{3\pi}{2}$.

Since $\cos u$ is negative in this range, $LN - M^2$ is negative. So all points in the region are hyperbolic. The points on the torus corresponding to these values of u are at a distance less than b from the axis of rotation. Hence the points on the torus whose distances are less than b are points which are hyperbolic points and these are inside points obtained by rotating BDA.

Sec: 2 Principal Curvature

Defn: The directions at P in which normal curvature has maximum or minimum values are called Principal directions at P. The Principal curvature at P are the normal curvatures at P along the Principal directions. and denote them by K_a & K_b.

Defn: If K_a and K_b are the Principal curvatures at a point P on the surface, then the mean curvature denoted by μ is defined as $\mu = \frac{1}{2}(K_a + K_b)$

$$\text{ie) } \mu = \frac{1}{2}(K_a + K_b) = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

Defn: 3 If K_a and K_b are Principal curvatures, the Gaussian curvature denoted by K is defined as $K = K_a K_b$

$$\text{ie) } K = K_a K_b = \frac{LN - M^2}{EG - F^2}$$

Defn: When the values of E, F, G and L, M, N are proportional, the principal directions are indeterminate and the normal curvature is the same in all directions. Such a point where $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$ is called an umbilic.

Note: At a point which is not an umbilic the two directions determined by eqn are orthogonal

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Lines of Curvature

Defn: A curve on a given surface whose tangent at each point is along a principal direction is called a line of curvature.

Thm: A necessary and sufficient condition for a curve on a surface to be a line of curvature is $\kappa d\bar{r} + d\bar{N} = 0$ at each point of the line of curvature where κ is the normal curvature in the direction of the line of curvature.

Proof: To prove the necessity of the condition:

Let the given curve on the surface be a line of curvature. Then the direction (du, dv) at any point P on the curve is a principal direction at (u, v) on the surface given by

$$(L - \kappa E)du + (M - \kappa F)dv = 0 \longrightarrow (1)$$

$$(M - \kappa F)du + (N - \kappa G)dv = 0 \longrightarrow (2)$$

where κ is one of the principal curvature at (u, v).

Let us write (1) & (2) by grouping the fundamental coefficients of the same kind

$$(L\,du + M\,dv) - \kappa(E\,du + F\,dv) = 0 \longrightarrow (3)$$

$$(M\,du + N\,dv) - \kappa(F\,du + G\,dv) = 0$$

Using the values $L = -N_1 \cdot r_1$, $M = -N_2 \cdot r_1$, $E = \bar{r}_1^2$, $G = \dots$

in the first eq of (3),

$$-(N_1\,du + N_2\,dv) \cdot \bar{r}_1 = \kappa(\bar{r}_1\,du + \bar{r}_2\,dv) \cdot \bar{r}_1 \longrightarrow (4)$$

Now $d\bar{r} = \frac{\partial \bar{r}}{\partial u} du + \frac{\partial \bar{r}}{\partial v} dv = \bar{r}_1 du + \bar{r}_2 dv \rightarrow (5)$

$d\bar{N} = \frac{\partial \bar{N}}{\partial u} du + \frac{\partial \bar{N}}{\partial v} dv = \bar{N}_1 du + \bar{N}_2 dv \rightarrow (6)$

Using (5) & (6) in (4), $(d\bar{N} + K d\bar{r}) \cdot \bar{r}_1 = 0 \rightarrow (7)$

IIIly $M = -\bar{N}_1 \cdot \bar{r}_2$ $N = -\bar{N}_2 \cdot \bar{r}_2$ $F = \bar{r}_1 \cdot \bar{r}_2$ $G = \bar{r}_2^2$

in the second eqn of (3) and Using (5) & (6).

$$(d\bar{N} + K d\bar{r}) \cdot \bar{r}_2 = 0 \rightarrow (8)$$

Claim $d\bar{N} + K d\bar{r} = 0$

Since $\bar{N}$ is a vector of constant modulus, $\bar{N}^2 = 1$ so that $\bar{N} \cdot d\bar{N} = 0$

$\therefore d\bar{N}$ is perpendicular to $\bar{N}$.

ie) $d\bar{N}$ is tangential to the surface.

Since $d\bar{N}$ and $d\bar{r}$ are tangential to the surface,

$(K d\bar{r} + d\bar{N})$ is also tangential to the surface.

Hence it lies in the planes of vectors $\bar{r}_1$ & $\bar{r}_2$.

from (7) & (8), we conclude that $K d\bar{r} + d\bar{N}$ is perpendicular to $\bar{r}_1$ and $\bar{r}_2$.

$\therefore K d\bar{r} + d\bar{N}$ is IIel to $\bar{r}_1 \times \bar{r}_2$ which is the direction of the surface normal.

Hence $K d\bar{r} + d\bar{N}$ is IIel to the surface normal contradicting the fact that $K d\bar{r} + d\bar{N}$ is tangential to the surface. This contradiction proves that

$$K d\bar{r} + d\bar{N} = 0$$

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To prove the sufficiency of the condition.

Let us assume that there is a curve on the surface for which $k d\bar{r} + d\bar{N} = 0$ for some fun. k at a point P on the surface.

Show that this curve is a line of curvature on the surface having k for its normal curvature at P.

To Prove this, it is enough if we show that at each point of the curve, its direction coincides with the principal direction.

Since $k d\bar{r} + d\bar{N} = 0$,

$(k d\bar{r} + d\bar{N}) \cdot \bar{r}_1 = 0$, $(k d\bar{r} + d\bar{N}) \cdot \bar{r}_2 = 0$

Hence if (du, dv) is the direction of the curve at a point (u, v), then by retracing the steps,

$$(L - kE)\,du + (M - kF)\,dv = 0$$

$$(M - kF)\,du + (N - kG)\,dv = 0$$

so that the direction at (u, v) coincides with the principal direction.

Further since $k d\bar{r} + d\bar{N} = 0$, we have $k d\bar{r} = -d\bar{N}$

Substituting for $d\bar{r}$ and $d\bar{N}$ from (5) & (6), we get

$$k(\bar{r}_1 du + \bar{r}_2 dv) = -(\bar{N}_1 du + \bar{N}_2 dv) \longrightarrow (9)$$

Taking dot product with $\bar{r}_1 du + \bar{r}_2 dv$ on both sides of (9)

we have $k(\bar{r}_1 du + \bar{r}_2 dv) \cdot (\bar{r}_1 du + \bar{r}_2 dv)$

$$= -(\bar{N}_1 du + \bar{N}_2 dv) \cdot (\bar{r}_1 du + \bar{r}_2 dv)$$

Using the fundamental coefficients, E, F, G & L, M, N

$$k(E du^2 + 2F du dv + G dv^2) = L du^2 + 2M du dv + N dv^2$$

$$K = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

which proves that K is the normal curvature of the curve at P in the direction (du, dv)

Since (du, dv) gives the principal directions at P

K is the ~~principal~~ normal curvature at P.

$\therefore$ The direction at each point of the curve is the principal direction having K for its normal curvature ~~in the principal direction~~.

Hence the curve must be a line of curvature on the surface and this completes the proof of Rodrigue's formula.

<u>Thm:</u> A necessary and sufficient condition that the lines of curvature be the parametric curves is that $F = 0$, $M = 0$.

<u>Proof:</u> Let the lines of curvature be taken as parametric curves.

Since they are orthogonal, $F = 0$.

Since $F = 0$, $H^2 = EG > 0 \Rightarrow E \neq 0 \;\; G \neq 0$

The differential eqn. of the lines of curvature is

$$(EM - FL)\,du^2 + (EN - GL)\,du\,dv + (FN - GM)\,dv^2 = 0 \quad \to (1)$$

The differential eqn. of the parametric curves is

$$du\,dv = 0 \longrightarrow (2)$$

From (1) & (2), $EM - FL = 0$ and $FN - GM = 0$

Since $F = 0$, $E > 0$, $G > 0$, we get $M = 0$.

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Thus $F=0$, $M=0$ are necessary conditions for the lines of curvature to be parametric curves.

Conversely, if $F=0$, $M=0$, the differential eqn of the lines of curvature (1) reduces to

$$(EN-GL)\,du\,dv=0$$

Since $EN-GL\neq 0$, $du\,dv=0$ so that the lines of curvature become parametric curves.

Thm: If K is the normal curvature in a direction making an angle ψ with the principal direction $v=$constant then $K=K_a\cos^2\psi+K_b\sin^2\psi$ where K_a & K_b are principal curvatures at the point P on the surface.

Proof: Since $El^2+2Flm+Gm^2=1$, the normal curvature at any point P in the direction (l,m) is

$$K=Ll^2+2Mlm+Nm^2 \longrightarrow (1)$$

Let us choose the lines of curvature at P as parametric curves.

Then $M=0$, $F=0$ so that (1) becomes

$$K=Ll^2+Nm^2 \longrightarrow (2)$$

The direction coefficients of the parametric curves $v=$const and $u=$constant are $\left(\frac{1}{\sqrt{E}},0\right)$ and $\left(0,\frac{1}{\sqrt{G}}\right)$

If K_a & K_b are the normal curvatures along these ~~Princip~~ principal directions, then from (2),

$$K_a=L\left(\frac{1}{\sqrt{E}}\right)^2=\frac{L}{E}$$

$$K_b=N\left(\frac{1}{\sqrt{G}}\right)^2=\frac{N}{G}$$

If φ is the angle between the given direction (l,m)

and the principal direction $\left(\frac{1}{\sqrt{E}}, 0\right)$ using the cosine formula $\cos\psi = Ell' + F(lm' + l'm) + Gmm'$

$$\cos\psi = El\frac{1}{\sqrt{E}} = l\sqrt{E} \quad (F=0) \quad \text{so that } l = \frac{\cos\psi}{\sqrt{E}}$$

Since the parametric curves are orthogonal, the angle between the direction (l, m) and $\left(0, \frac{1}{\sqrt{G}}\right)$ is $90-\psi$

Hence $\cos(90-\psi) = Gm\frac{1}{\sqrt{G}} = m\sqrt{G}$

$$\sin\psi = m\sqrt{G}$$

$$m = \frac{\sin\psi}{\sqrt{G}}$$

$\therefore$ (2) becomes,

$$K = \frac{L}{E}\cos^2\psi + \frac{N}{G}\sin^2\psi$$

Since $K_a = \frac{L}{E}$ and $K_b = \frac{N}{G}$, we get

$$K = K_a\cos^2\psi + K_b\sin^2\psi.$$

The Dupin indicatrix.

Let O be any ~~given~~ point on the given surface. Then the section of the surface by a plane parallel to the tangent plane at O on it and at a very small distance from it is called the Dupin indicatrix at O.

If K_a, K_b are the principal curvatures at O on a surface, then the equation of the Dupin indicatrix is $\frac{x^2}{R_a} + \frac{y^2}{R_b} = 2h$, $z = h$ where $R_a = \frac{1}{K_a}$ & $R_b = \frac{1}{K_b}$

Note! Since Dupin indicatrix is a conic section, the nature of the conic section depends on the point O on the surface.

If k_a, k_b have the same sign, the conic is an ellipse with semi-axes of length $(2hR_a)^{1/2}, (2hR_b)^{1/2}$ and is real or imaginary according to the sign of h.

If k_a, k_b have different signs, the conic is one of two conjugate hyperbolas according to the sign of h. In this case the directions of the asymptotes at O are called the asymptotic directions at O. The normal curvature changes sign as the direction passes through an asymptotic position.

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4. Developables: 1. The envelope of one-parameter family of planes is called a developable surface or a developable. Such a family of plane is given by the eqn. $\bar{r} \cdot \bar{a} = p$, where $\bar{a}$ is a vector ~~nl.~~ to the plane & p is the ~~+r~~ distance from the origin to the plane and p ~~are~~ functions of a real parameter u and $\bar{r}$ is independent of u.

Defn! 2 The line of intersection of the two consecutive planes is called the Characteristic line. The ~~characteri~~ Characteristic line corresponding to the plane u are given by the intersection of the planes $\bar{r} \cdot a = p$ and $\bar{r} \cdot \dot{\bar{a}} = \dot{p}$

Defn: 3 When the planes $f(u) = 0$, $f(v)$ and $f(w) = 0$ intersect at a point, the limiting position of the point of intersection of the 3 planes as $v \to u$ & $w \to u$ is called the characteristic point corresponding to the plane u.

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The characteristic point of the plane u is determined by the equations $\bar{r}\cdot\bar{a}=p$, $\bar{r}\cdot\dot{\bar{a}}=\dot{p}$, $\bar{r}\cdot\ddot{\bar{a}}=\ddot{p}$

Defn: The locus of the characteristic points is called the edge of regression of the developable.

Thm: A developable consists of two sheets which ... tangents to the edge of regression along the sharp edge.

Proof: Let O be the point $s=0$ on the edge of regression curve C and choose the orthogonal triad at O as the rectangular co-ordinate axes $O(x,y,z)$. If R is the position vector of the point $P(x,y,z)$ on the developable with respect to O, then we have

$$R=x\bar{t}+y\bar{n}+z\bar{b} \rightarrow (1)$$

The position vector of any point on the developable surface is $R(s,v)=\bar{r}(s)+v\bar{t}(s) \rightarrow (2)$

Let us expand $\bar{r}(s)$ and $\bar{t}(s)$ of (2) in power series of s by Taylor series at the origin O.

Then we get $R=s\frac{d\bar{r}}{ds}+\frac{s^2}{2}\frac{d^2\bar{r}}{ds^2}+\frac{s^3}{3!}\frac{d^3\bar{r}}{ds^3}+o(s^4)$

$$+v\left\{\bar{t}+s\frac{d\bar{t}}{ds}+\frac{s^2}{2}\frac{d^2\bar{t}}{ds^2}+o(s^3)\right\} \rightarrow (3)$$

Using Serret-Frenet formulae,

$$\bar{r}(s)=s\bar{t}+\frac{s^2}{2}\bar{t}'+\frac{s^3}{6}(k'\bar{n}+k\bar{n}')+o(s^4)$$

$$=s\bar{t}+\frac{s^2}{2}k\bar{n}+\frac{s^3}{6}(k'\bar{n}+k\tau\bar{b}-k^2\bar{t})+o(s^4)$$

$$v\bar{t}=v\left\{\bar{t}+sk\bar{n}+\frac{1}{2}s^2(k'\bar{n}+k\tau\bar{b}-k^2\bar{t})+o(s^3)\right\}$$

$\therefore$ (3)

$$R(s,v) = s\bar{t} + \frac{1}{2}s^2 k\bar{n} + \frac{1}{6}s^3\{k'\bar{n} + k\tau\bar{b} - k^2\bar{t}\} + O(s^4)$$
$$+ v\{\bar{t} + sk\bar{n} + \frac{1}{2}s^2(k'\bar{n} + k\tau\bar{b} - k^2\bar{t}) + O(s^3)\} \to (4)$$

Now equating the coeff. of $\bar{t}$ in (1) & (4),

$$x = s - \frac{1}{6}s^3k^2 + O(s^4) + v\{1 - \frac{s^2}{2}k^2 + O(s^3)\} \to (5)$$

Let us find the point of intersection of the developable with the normal plane $x=0$ at O. The value of the parameter of the point of intersection of the normal plane with the surface is obtained from $x=0$ in (5)

Hence when $x=0$, we have from (5),

$$v = -\left(s - \frac{1}{6}s^3k^2\right)\left(1 - \frac{s^2}{2}k^2\right)^{-1}$$
$$= -\left(s - \frac{1}{6}s^3k^2\right)\left(1 + \frac{s^2}{2}k^2 + \cdots\right)$$
$$= -s - \frac{s^3k^2}{3} + O(s^4)$$

Substituting the above value of v in (3), we get the position vector of the point of intersection of the surface and the normal plane at O. Then we have

$$R(s,v) = s\bar{t} + \frac{1}{2}s^2k\bar{n} + \frac{1}{6}s^3(k'\bar{n} + k\tau\bar{b} - k^2\bar{t}) + O(s^4)$$
$$+ \left(-s - \frac{1}{3}s^3k^2 + O(s^4)\right)\left[\bar{t} + sk\bar{n} + \frac{1}{2}s^2(k'\bar{n} + k\tau\bar{b} - k^2\bar{t})\right] \to (6)$$

Equating the coefficients of $\bar{t}, \bar{n}, \bar{b}$ in (1) & (6),

$$x = 0 \quad y = -\frac{1}{2}s^2k + O(s^3) \quad z = -\frac{1}{3}k\tau s^3 + O(s^4)$$

which is a curve in the normal plane.

Eliminating s we get $\frac{z^2}{y^3} = -\frac{8}{9}\frac{\tau^2}{k}$

$$\therefore z^2 = -\frac{8}{9}\frac{\tau^2}{k}y^3$$

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This shows that the intersection of the developable with the normal plane of the edge of regression has a cusp whose tangent is along $y=0$ & $z=0$ which the principal normal.

Thus two sheets of the developable surface are thus tangent to the edge of regression along a sharp edge.

Sec: 5 Developables associated with space curves.

Defn: The envelope of the family of osculating planes of a space curve is called an osculating developables. Its characteristics lines are tangents to the curve and hence this developable is also called the tangential developable.

Thm: The space curve is itself the edge of regression of the osculating developable.

Proof: Let $\bar{r} = \bar{r}(s)$ be the equation of the space curve and $\bar{R}$ be the position vector of any point on the osculating plane.

Then the equation of the osculating plane is

$$f(s) = (R-\bar{r})\cdot\bar{b} = 0 \longrightarrow (1)$$

Since $\bar{r}$ and $\bar{b}$ are functions of s, (1) is a single parameter family of planes.

To find the equation of the osculating developables

Let us differentiate (1) w.r.to 's'

$$f'(s) = (R-\bar{r})\cdot\bar{b}' - \bar{r}'\cdot\bar{b} = 0$$

$$(R-\bar{r})\cdot(-\tau\bar{n}) - \bar{t}\cdot\bar{b} = 0$$

Since $t \cdot b = 0$, & $t \neq 0$, we get $(R-\bar{\gamma})\cdot\bar{n} = 0 \longrightarrow (2)$
which is the rectifying plane at $\bar{\gamma} = \bar{\gamma}(s)$ of the curve.
Since the characteristic line is the intersection of (1) & (2), it is the tangent to C at $\bar{\gamma} = \bar{\gamma}(s)$.
Thus the characteristic lines to the osculating developables are tangents to C.

To find the edge of regression.

Let us differentiate (2), w.r.to s.

Then $(R-\bar{\gamma})\cdot\bar{n}' - \bar{\gamma}'\cdot\bar{n} = 0$

(i.e.) $(R-\bar{\gamma})\cdot(\tau\bar{b} - K\bar{t}) - \bar{t}\cdot\bar{n} = 0$

Since $K \neq 0$ & $\bar{t}\cdot\bar{n} = 0$ using (1) in the last eqn,

$$(R-\bar{\gamma})\cdot\bar{t} = 0$$

which is the normal plane at $\bar{\gamma} = \bar{\gamma}(s)$ to C.

The point of intersection (1), (2) & (3) is the characteristic point and its locus is the edge of regression.

Since (1), (2) & (3) intersect at $\bar{\gamma} = \bar{\gamma}(s)$ on the curve, the characteristic point is the curve is the characteristic every point $\bar{\gamma} = \bar{\gamma}(s)$ of the curve coincide with point so that the characteristic points coincide with the edge of regression every point on the curve. Hence the edge of regression is the given curve. ——.

Defn: The envelope of the normal planes to the space curve is called the polar developable.

Thm: The edge of regression of the polar developable of a space curve is the locus of the centres of spherical curvature.

Proof: Let $\bar{r} = \bar{r}(s)$ be the given space curve and $\bar{R}$ be any point on the normal plane. Then the equation of the normal plane is

$$(R - \bar{r}) \cdot \bar{t} = 0 \longrightarrow (1)$$

Since $\bar{r}$ and $\bar{t}$ are functions of a single parameter, (1) is a single parameter family of planes whose envelope is the polar developable.

To find the edge of regression, we shall find the characteristic point from the following equations.

Differentiating (1) w.r.to 's' we have

$$(R - \bar{r}) \cdot \bar{t}' - \bar{r}' \cdot \bar{t} = 0$$

$$(R - \bar{r}) \cdot K\bar{n} - 1 = 0 \quad [\because \bar{t} \cdot \bar{t} = 1]$$

$$\therefore (R - \bar{r}) \cdot \bar{n} = \frac{1}{K} = \rho \longrightarrow (2)$$

Differentiating (2) w.r.to 's' we get

$$(R - \bar{r})\bar{n}' - \bar{r}' \cdot n = \rho'$$

$$(R - \bar{r})(\tau\bar{b} - K\bar{t}) - \bar{t} \cdot \bar{n} = \rho'$$

$$(R - \bar{r}) \cdot \bar{b} = \frac{\rho'}{\tau} = \rho'\sigma \quad [\because \bar{t} \cdot \bar{n} = 0 \text{ \& by (1)}]$$

$$\longrightarrow (3)$$

The point of intersection of (1), (2) & (3) is the characteristic point and its locus is the edge of regression of the polar developables.

Since $(R - \bar{r})$ is orthogonal to $\bar{t}$, $(R - \bar{r})$ lies in the plane of $\bar{n}$ & $\bar{b}$ so that we can take

$$R - \bar{r} = \lambda\bar{n} + \mu\bar{b} \quad \text{or} \quad R = \bar{r} + \lambda\bar{n} + \mu\bar{b}$$

$$\longrightarrow (4)$$

Now we determine the scalars λ & μ.
Taking dot product with $\bar{n}$ on both sides of (6) we get

$$(\bar{R} - \bar{r}) \cdot \bar{n} = \lambda \quad \text{——(8)}$$

Comparing (2) & (8), we obtain $\lambda = \rho$.
Taking dot product with $\bar{b}$ on both sides of (6)

$$(\bar{R} - \bar{r}) \cdot \bar{b} = \lambda \bar{n}\cdot\bar{b} + \mu \bar{b}\cdot\bar{b}$$

Since $\bar{n}\cdot\bar{b} = 0$, $\bar{b}\cdot\bar{b} = 1$, using eqn (3), we have $\mu = \sigma\rho'$
Substituting the values of λ & μ in (6), we get

$$\bar{R} = \bar{r} + \rho\bar{n} + \sigma\rho'\bar{b} \quad \text{——(9)}$$

which gives the position vector of the characteristic point. But we know that $\bar{R}$ gives the position vector of the centre of spherical curvature. Thus the characteristic point of the polar developable coincides with the centre of spherical curvature. Hence the edge of regression of the polar developable is the locus of the centre of spherical curvature.

Thm: A necessary and sufficient condition for a surface to be a developable is that its Gaussian curvature shall be zero.

Proof: To prove the necessity of the condition let us assume that the surface is a developable and show that its Gaussian curvature is zero.

If the developable is a cylinder or a cone, the Gaussian curvature is zero, since at each point on the cone or cylinder one of the principal directions is the generating straight line whose curvature is zero. Hence excluding these two cases of developables.

We are left with developables in general.

Since a developable can be considered as the osculating developable of its edge of regression, it is generated by the tangents to the edge of regression.

Let $\bar{r} = \bar{r}(s)$ be the eqn of the edge of regression on the developable.

If P is any point on this curve, then the tgt at P is the characteristic line of the developable.

Since the characteristic lines generates the developable surface, any point Q with the position vector R on the tangent is a point on the developable surface so that it can be taken as

$$R(s, v) = \bar{r}(s) + v\bar{t}(s)$$

For this surface, let us find E, F, G, L, M & N and show that its Gaussian curvature $K = \dfrac{LN - M^2}{EG - F^2} = 0$

$$\bar{R}_1 = \frac{\partial \bar{R}}{\partial s} = \frac{d\bar{r}}{ds} + v\frac{d\bar{t}}{ds} = \bar{t} + vk\bar{n}$$

$$\bar{R}_2 = \frac{\partial \bar{R}}{\partial v} = \bar{t}(s)$$

$$\bar{R}_{11} = \frac{d\bar{t}}{ds} + vk'\bar{n} + vk\bar{n}' = k\bar{n} + vk'\bar{n} + vk(\tau\bar{b} - k\bar{t})$$

$$\bar{R}_{12} = \bar{R}_{21} = k\bar{n} \quad R_{22} = 0.$$

$$E = 1 + v^2k^2 \quad F = (t + vkn) \cdot t = 1 \quad G = \bar{t} \cdot \bar{t} = 1$$

$$H^2 = EG - F^2 = v^2k^2$$

$$N = \frac{\bar{R}_1 \times \bar{R}_2}{H} = -\frac{vk\bar{b}}{vk} = -\bar{b}$$

$$L = N \cdot R_{11} = -b \cdot [k\bar{n} + vk'\bar{n} + vk(\tau\bar{b} - k\bar{t})]$$

$$= -vk\tau$$

$M = N \cdot R_{12} = -b \cdot (k\bar{n}) = 0$

$N = N \cdot R_{22} = 0$

Hence using the above values, the Gaussian curvature

$$K = \frac{LN - M^2}{EG - F^2} = 0$$

and this proves the necessity condition.

To prove the converse

Let us assume that $K = 0$ for a surface $\bar{r} = \bar{r}(u,v)$ and show that it is developable surface.

To this end, we have to show that $\bar{r} = \bar{r}(u,v)$ is generated by a single parameter family of planes.

Since $L = -\bar{r}_1 \cdot N_1$, $M = -\bar{r}_1 \cdot N_2 = -N_1 \cdot \bar{r}_2$, $N = -\bar{r}_2 \cdot N_2$

$$LN - M^2 = (\bar{r}_1 \cdot N_1)(\bar{r}_2 \cdot N_2) - (\bar{r}_1 \cdot N_2)(\bar{r}_2 \cdot N_1)$$

$$= (\bar{r}_1 \times \bar{r}_2) \cdot (\bar{N}_1 \times \bar{N}_2)$$

Since $\bar{r}_1 \times \bar{r}_2 = H\bar{N}$, we have $LN - M^2 = HN \cdot (\bar{N}_1 \times \bar{N}_2)$

$$= H[N\, N_1\, N_2]$$

As $H \neq 0$, $K = 0$ implies that $LN - M^2 = 0$ so that $[N, N_1, N_2] = 0$

Since $\bar{N} \cdot \bar{N} = 1$, we have $\bar{N} \cdot \bar{N}_1 = 0$ and $\bar{N} \cdot \bar{N}_2 = 0$

So, N is perpendicular to both $\bar{N}_1$ and $\bar{N}_2$.

Hence $\bar{N}$ is parallel to $\bar{N}_1 \times \bar{N}_2$.

Thus $[\bar{N}, \bar{N}_1, \bar{N}_2] = \bar{N} \cdot (\bar{N}_1 \times \bar{N}_2)$ cannot be zero unless $N_1 = 0$ or $N_2 = 0$ or N_1 is parallel to N_2.

So $[\bar{N}, N_1, \bar{N}_2] = 0$ under the given condition implies the follo. two cases.

(i) $\bar{N}_1 = 0$ or $\bar{N}_2 = 0$ (ii) $\bar{N}_1 = \mu \bar{N}_2$

Case(i) Let us consider $\bar{N}_2 = 0$ only, since a similar argument is true for $\bar{N}_1 = 0$.

The equation of the tangent plane at $\bar{r} = \bar{r}(u,v)$ on the surface is $(\bar{R} - \bar{r}) \cdot \bar{N} = 0 \longrightarrow (2)$

where $\bar{R}$ is the position vector of any point on the tangent plane and $\bar{R}$ is indep. of u, v.

Differentiating (2) partially w.r.to 'v', we get

$$\frac{\partial}{\partial v}[(\bar{R}-\bar{r})\cdot\bar{N}] = -\bar{r}_2 \cdot \bar{N} + (\bar{R}-\bar{r})\cdot\bar{N}_2 \longrightarrow (3)$$

Now $\bar{N}_2 = 0$ and since $\bar{r}_2$ is tangential to the surface, $\bar{r}_2 \cdot \bar{N} = 0$.

Hence eqn (3) gives $\frac{\partial}{\partial v}[(\bar{R}-\bar{r})\cdot\bar{N}] = 0$

so that $(\bar{R}-\bar{r})\cdot\bar{N}$ is indep. of v.

Thus the eqn. of the tangent plane contains only one parameter u.

Thus the surface is the envelope of a single parameter family of planes and hence it is a developable.

Case(ii) In this case, when $\bar{N}_1 = \mu\bar{N}_2$, let us consider a suitable change of parameter from u, v to u', v' so that we can use the previous case. Let the transformation be

$$u = u' + v' \text{ and } v = u' - \mu v'$$

This is a proper parametric transformation which preserves surface normal N.

Now $\bar{N}_1' = \dfrac{\partial \bar{N}}{\partial u'} = \dfrac{\partial \bar{N}}{\partial u} \cdot \dfrac{\partial u}{\partial u'} + \dfrac{\partial \bar{N}}{\partial v} \cdot \dfrac{\partial v}{\partial u'}$

$$= \bar{N}_1 + \bar{N}_2 \neq 0.$$

$$N_2' = \frac{\partial N}{\partial v'} = \frac{\partial \bar{N}}{\partial u} \cdot \frac{\partial u}{\partial v'} + \frac{\partial N}{\partial v} \cdot \frac{\partial v}{\partial v'}$$

$$= N_1 - \mu N_2.$$

After the transformation $\bar{N}_1'$ and $\bar{N}_2'$ are not ||el as seen from the above equation.

Since $\bar{N}_2' = 0$, as in the previous case, the tangent plane at P is a single parameter family of planes so that the given surface is a developable surface.

Sec:6 __Monge's theorem:__

__Thm:__ A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

__Proof:__ Let $\bar{r} = \bar{r}(s)$ be a curve lying on the surface $\bar{r} = \bar{r}(u, v)$. Let $\bar{N}$ be the unit surface normal at a point $\bar{r} = \bar{r}(s)$ on the curve so that $\bar{N}$ can be considered as a function of s only.

We shall prove the theorem in the following two steps.

__Step:1)__ In this step, we prove that the normals to the surface $\bar{r} = \bar{r}(u, v)$ along the curve $\bar{r} = \bar{r}(s)$ form a developable iff $[\bar{t}, \bar{N}, \bar{N}'] = 0$.

If $\bar{R}$ is the position vector of any point Q on the surface normal along the curve, then R is a point on the developable generated by the normals and it can be taken as $R(s,v) = \bar{r}(s) + v\bar{N}(s)$

where v is the distance PQ along the surface normal at P.

The surface generated by the surface normals is a developable iff its Gaussian curvature is zero at every point. This implies that $LN - M^2 = 0$ at every point of the surface.

Hence, let us find L, M, N for the surface.

$$\bar{R}_1 = \frac{\partial \bar{R}}{\partial s} = \frac{d\bar{r}}{ds} + v\frac{d\bar{N}}{ds}$$

$$= \bar{t} + v\bar{N}', \quad \bar{R}_{11} = k\bar{n} + v\bar{N}''$$

$$\bar{R}_2 = \frac{d\bar{R}}{dv} = \bar{N} \quad \bar{R}_{12} = \bar{R}_{21} = N', \quad \bar{R}_{22} = 0.$$

Now, $L = R_{11} \cdot \bar{N} = [k\bar{n} + vN''] \cdot \bar{N} \neq 0$.

$$HM = [R_{12}, R_1, R_2] = [N', \bar{t} + vN', N]$$

$$HM = [N', t, N] + v[N', N', N]$$

Since $\bar{N}'$ is equal to N', $[N', N', N] = 0$

As $H \neq 0$, $M = \frac{1}{H}[\bar{N}', \bar{t}, \bar{N}] = \frac{1}{H}(\bar{t}, \bar{N}, \bar{N}')$

Since $R_{22} = 0$ & $H \neq 0$, $HN = [R_{22}, R_1, R_2] = 0$

So that $N = 0$.

As $L \neq 0$ & $N = 0$, $LN - M^2 = 0$ iff $M = 0$ which gives $[\bar{t}, \bar{N}, \bar{N}'] = 0$

Step:2 To prove the theorem, it is enough if we show that $[\bar{t}, \bar{N}, \bar{N}'] = 0$ is the necessary and sufficient condition for $\bar{r} = \bar{r}(s)$ to be a line of curvature on the surface $\bar{r} = \bar{r}(u, v)$.

To prove the necessity of the condition

Let us assume that $\bar{r} = \bar{r}(s)$ is a line of curvature on $\bar{r} = \bar{r}(u, v)$. Then we have by Rodrigues formula

$k\frac{d\bar{r}}{ds} + \frac{d\bar{N}}{ds} = 0$ which gives $\bar{N}' = k\bar{t}$

$$\therefore [\bar{t}, \bar{N}, \bar{N}'] = [\bar{t}, \bar{N}, k\bar{t}] = 0.$$

Hence the surface normals along the curve $\bar{r} = \bar{r}(s)$ form a developable surface.

Conversely, assuming $[\bar{t}, \bar{N}, \bar{N}'] = 0$ show that $\bar{r} = \bar{r}(s)$ is a line of curvature on the surface.

Now $[\bar{t}, \bar{N}, \bar{N}'] = 0 \Rightarrow [\bar{t}, \bar{N}', \bar{N}] = 0$

$\Rightarrow (\bar{t} \times \bar{N}') \cdot N = 0$

Further $\bar{N} \neq 0$, and $\bar{N}^2 = 1$ so that $\bar{N} \cdot \bar{N}' = 0$

$\Rightarrow \bar{N}'$ is $\perp$r to $\bar{N}$.

$\Rightarrow \bar{N}'$ is in the tangent plane at P.

b) $\bar{N}'$ is in the $\parallel$el to N.

Hence $\bar{t} \times \bar{N}'$ is $\parallel$el to $\bar{N}$

So, if $\bar{t} \times \bar{N}' \neq 0$, $(\bar{t} \times N') \cdot \bar{N}$ cannot be zero because two non-zero parallel vectors $\bar{t} \times \bar{N}'$ & $\bar{N}$ will have scalar product different from zero.

$\therefore (\bar{t} \times \bar{N}') \cdot \bar{N} = 0 \rightarrow \bar{t} \times \bar{N} = 0$ which is true iff one vector is a scalar multiple of the other.

So, we can take $\bar{N}' = -K\bar{t}$ ...

Now $\bar{N}' = -K\bar{t}$ gives $\frac{d\bar{N}}{ds} = -K\frac{d\bar{r}}{ds}$

(i.e.) $d\bar{N} + K d\bar{r} = 0$ which is the Rodrigue formula characterising the lines of curvature.

Thus $\bar{r} = \bar{r}(s)$ is a line of curvature on the surface.

Sec 7 Minimal Surfaces:

Defn: If the mean curvature $\mu = \frac{1}{2}(K_a + K_b)$ is zero at all points of a surface, then the surface is called a minimal surface.

Note: From the defn. $\mu = \frac{1}{2}(K_a + K_b) = \frac{EN + GL - 2FM}{2(EG - F^2)}$

Since $EG - F^2 \neq 0$, the condition for a minimal surface becomes $EN + GL - 2FM = 0$.

When the parametric curves are orthogonal, the above condition reduces to $EN + GL = 0$ $[\because \bar{r}_1 \cdot \bar{r}_2 = 0$ (i.e.) $F = 0]$

Thm: If there is a surface of minimum area passing through a closed curve, then it is necessarily a minimal surface in the sense that it is of zero mean curvature.

Proof: Let $\bar{r} = \bar{r}(u, v)$ be a surface bounded by a closed curve C. Let S' be another surface obtained from S by a small displacement ϵ in the direction of N where ϵ is a function of u, v.

Further we take $\frac{\partial \epsilon}{\partial u} = \epsilon_1$ and $\frac{\partial \epsilon}{\partial v} = \epsilon_2$ are small and satisfies the condition $\epsilon_1 = O(\epsilon)$ and $\epsilon_2 = O(\epsilon)$ as $\epsilon \to 0$

Since S' is obtained after a small displacement along ..., the position vector of any point on S' is given by

$$R(u,v) = \bar{r}(u,v) + \epsilon \bar{N} \longrightarrow (1)$$

where $\bar{r}, \epsilon$ & $\bar{N}$ are functions of u & v.

If A is the area of the surface S, we prove the theorem from the first variation δA which we find as follows.

Let E^*, F^*, G^* denote the first fundamental coefficients of S' we find $A^* = \int_S H^* du\,dv$ where $H^{*2} = E^* G^* - F^{*2}$

Differentiating (1) partially w.r.to u and v we get

$$\bar{R}_1 = \bar{r}_1 + \epsilon_1 \bar{N} + \epsilon \bar{N}_1$$

$$\bar{R}_2 = \bar{r}_2 + \epsilon_2 \bar{N} + \epsilon \bar{N}_2$$

$$E^* = \bar{R}_1 \cdot \bar{R}_1 = (\bar{r}_1 + \epsilon_1 \bar{N} + \epsilon \bar{N}_1) \cdot (\bar{r}_1 + \epsilon_1 \bar{N} + \epsilon \bar{N}_1)$$

$$= \bar{r}_1 \cdot \bar{r}_1 + 2\epsilon_1 \bar{r}_1 \cdot \bar{N} + 2\epsilon \bar{r}_1 \cdot \bar{N}_1 + O(\epsilon^2)$$

Since $\bar{r}_1 \cdot \bar{N} = 0$ & $L = -\bar{r}_1 \cdot \bar{N}_1$, we get

$$E^* = E - 2\epsilon L + O(\epsilon^2)$$

$$F^* = \bar{R}_1 \cdot \bar{R}_2 = (\bar{r}_1 + \epsilon_1 \bar{N} + \epsilon \bar{N}_1) \cdot (\bar{r}_2 + \epsilon_2 \bar{N} + \epsilon \bar{N}_2)$$

$$= \bar{r}_1 \cdot \bar{r}_2 + \epsilon_2 \bar{r}_1 \cdot \bar{N} + \epsilon \bar{r}_1 \cdot \bar{N}_2 + \epsilon_1 \bar{N} \cdot \bar{r}_2 + \epsilon \bar{N}_1 \cdot \bar{r}_2 + O(\epsilon^2)$$

Since $\bar{r}_1 \cdot \bar{N} = \bar{r}_2 \cdot \bar{N} = 0$, $M = -\bar{r}_1 \cdot \bar{N}_2 = -\bar{r}_2 \cdot \bar{N}_1$, we get

$$F^* = F - 2\epsilon M + O(\epsilon^2)$$

$$G^* = \bar{R}_2 \cdot \bar{R}_2 = (\bar{r}_2 + \epsilon_2 \bar{N} + \epsilon \bar{N}_2) \cdot (\bar{r}_2 + \epsilon_2 \bar{N} + \epsilon \bar{N}_2)$$

$$= G_2 - 2\epsilon N + O(\epsilon^2)$$

Now, $H^* = E^* G^* - F^{*2}$

$$= (E - 2\epsilon L + O(\epsilon^2))(G - 2\epsilon N + O(\epsilon^2)) - (F - 2\epsilon M + O(\epsilon^2))^2$$

$$= EG - F^2 - 2\epsilon(EN - 2FM + GL) + O(\epsilon^2)$$

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$$= H^2 - 2\epsilon H^2 \left[\frac{EN - 2FM + GL}{2H^2}\right] + O(\epsilon^2)$$

Since $\mu = \frac{EN - 2FM + GL}{2H^2}$ is the mean curvature

$$H^{*2} = H^2 - 2\mu\epsilon H^2 + O(\epsilon^2)$$

$$H^* = H(1 - 2\epsilon\mu + O(\epsilon^2))^{1/2}$$

$$= H(1 - 2\epsilon\mu) + O(\epsilon^2)$$

Let A & A^* denote the area of the surfaces

Then we know that $A = \int_S H\,du\,dv$

Now, $$A^* = \int_{S^*} H^*\,du\,dv = \int_{S^*} (H - 2\epsilon\mu H)\,du\,dv + O(\epsilon^2)$$

$$= \int_S H\,du\,dv - \int 2\epsilon\mu H\,du\,dv + O(\epsilon^2)$$

$$= A - \int_S 2\epsilon\mu H\,du\,dv + O(\epsilon^2)$$

$$A^* - A = -\int_S 2\epsilon\mu H\,du\,dv + O(\epsilon^2) \longrightarrow (1)$$

Let $A^* - A = \delta A$ & also we have $dA = H\,du\,dv$

Hence neglecting terms containing ϵ^2, equation (1) becomes $\delta A = -\int_S 2\epsilon\mu\,dA$ when $\epsilon \to 0$.

Hence we conclude that δA is the first variation of the area enclosed by the fixed curve C.

The first variation δA vanishes for all $\epsilon > 0$ iff $\mu = 0$. That is the mean curvature vanishes so that the surface is a minimal surface.