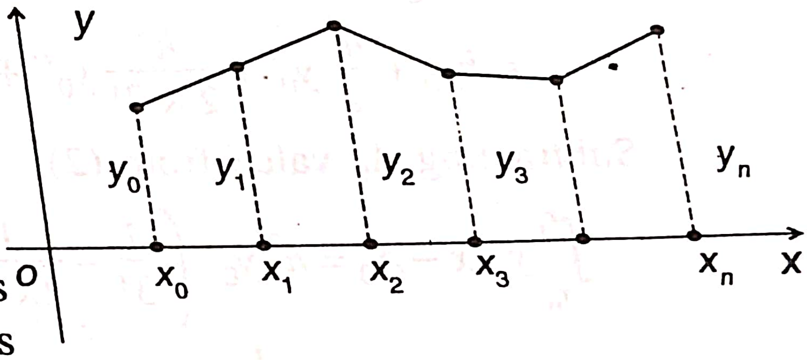
This is known as Trapezoidal Rule.

9·10. Geometrical interpretation

Geometrically, if the ordered pairs (x_i, y_i), $i = 0, 1, 2, \ldots n$ are plotted, and if any two consecutive points are joined by straight lines, we get the figure as shown.



The area between $f(x)$, x-axis and ordinates $x = x_0$ and $x = x_n$ is approximated to the sum of the trapeziums as shown in the figure.

Note: Though this method is very simple for calculation purposes of numerical integration, the error in this case is significant. The accuracy of the result can be improved by increasing the number of intervals and decreasing the value of h.

Richardson's method is called *Romberg method* or *Romberg integration.*

9·13. Simpson's one-third rule

Setting $n = 2$ in Newton-cote's quadrature formula, we have

$$\int_{x_0}^{x_2} f(x)\, dx \approx h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0\right]$$

(since other terms vanish)

$$\approx h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(E-1)^2 y_0\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[\frac{1}{3}y_2 + \frac{4}{3}y_1 + \frac{1}{3}y_0\right]$$

$$= \frac{h}{3}(y_2 + 4y_1 + y_0)$$

Similarly, $\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$

$$\int_{x_i}^{x_{i+2}} f(x)\,dx = \frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})$$

If n is an even integer, last integral will be

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals, if n is an even positive integer, that is, the number of ordinates $y_0, y_1, \ldots y_n$ is odd, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots) + 4(y_1 + y_3 + \cdots)]$$

$= \frac{h}{3}$ [sum of the first and last ordinates
+ 2 (sum of remaining odd ordinates)
+ 4 (sum of even ordinates)]

Note: Though y_2 has suffix even, it is the third ordinate (odd).

9·14. Simpson's three-eighths rule

Putting $n = 3$ in Newton-cotes formula (equation 2, §9·8)

$$\int_{x_0}^{x_3} f(x)\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(\frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4} - 27 + 9\right)\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(E-1)^2 y_0 + \frac{3}{8}(E-1)^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}y_1 - \frac{9}{2}y_0 + \frac{9}{4}(y_2 - 2y_1 + y_0)\right.$$

$$\left. + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}[y_3 + 3y_2 + 3y_1 + y_0]$$

If n is a multiple of 3,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{3h}{8}\Big[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots$$
$$+ (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\Big]$$

$$= \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1})$$
$$+ 2(y_3 + y_6 + y_9 + \cdots + y_n)\Big] \quad \ldots(2)$$

Equation (2) is called *Simpson's three-eighths rule* which is applicable only when n is a multiple of 3.

9.15. Weddle's rule

Putting $n = 6$ in Newton-cotes formula

$$\int_{x_0}^{x_0+6h} f(x)\,dx = h\,[6y_0 + 18\,\Delta y_0 + \frac{1}{2}(72 - 18)\,\Delta^2 y_0$$
$$+ \frac{1}{6}(324 - 216 + 36)\,\Delta^3 y_0 + \cdots]$$

$$= h\Big[6y_0 + 18\,\Delta y_0 + 27\,\Delta^2 y_0 + 24\,\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0$$
$$+ \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\Big]$$

Replace the term $\frac{41}{140}\Delta^6 y_0$ by $\frac{42}{140}\Delta^6 y_0$. By this change, the error introduced is only $\frac{h}{140}\Delta^6 y_0$ which is negligible when h and $\Delta^6 y_0$ are small. Using $\Delta = E - 1$ and replacing all differences in terms of y's, we get

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly,

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

....

$$\int_{x_0+(n-6)h}^{x_0+nh} f(x)\,dx = \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\,[(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5)$$
$$+ (2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11})$$
$$+ (\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots)$$
$$+ (2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n)] \quad \ldots(1)$$

Equation (1) is called *Weddle's rule.*

In the above formula, the coefficients may be remembered in groups of six.

First group : Coefficients : 1, 5, 1, 6, 1, 5

All interior groups : Coefficients : 2, 5, 1, 6, 1, 5

Last group : Coefficients : 2, 5, 1, 6, 1, 5, 1.

Note: If there are only 7 ordinates, the coefficients are 1, 5, 1, 6, 1, 5, 1.

Notes: 1. In trapezoidal rule, $y(x)$ is a linear function of x. The rule is the simplest one but it is least accurate.

2. In Simpson's one-third rule, $y(x)$ is a polynomial of degree *two*. To apply this rule, n, the number of intervals must be *even*. That is, the number of ordinates must be *odd*.

3. In Simpson's three-eighths rule, $y(x)$ is a polynomial of degree *three*. This rule is applicable if n, the number of intervals is a multiple of 3.

4. In Weddle's rule, $y(x)$ is a polynomial of degree *six* and this rule is applicable only if n, the number of intervals, is a multiple of *six*. A minimum number of 7 ordinates is necessary.